



TITLE:

# Boundary Values of Hyperfunction Solutions of Linear Partial Differential Equations (佐藤の超函 数とその応用)

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Boundary values of hyperfunction solutions of  
linear partial differential equations <sup>(\*)</sup>

By Hikosaburo KOMATSU and Takahiro KAWAI

Let  $P(x, D)$  be a linear partial differential operator with real analytic coefficients in a domain  $V$  in  $\mathbb{R}^{n+1}$  and let  $S \subset V$  be a real analytic hypersurface non-characteristic with respect to  $P(x, D)$ . The purpose of this paper is to show that every hyperfunction solution  $u$  of  $P(x, D)u = 0$  on one side of  $V \setminus S$  has boundary values on  $S$  which are hyperfunctions of  $n$  variables on  $S$ .

This fact has been proved by H. Komatsu [6] and P. Schapira [8] in the case where  $P(x, D)$  is elliptic. Their method applies with minor modifications to the general operators.

In §1 we show that the Cauchy-Kowalevsky theorem for the dual equation with the initial values on  $S$  is equivalent to a theorem of division of hyperfunctions with supports in  $S$  by the differential operator  $P(x, D)$ .

We define the boundary values in §2 and prove the uniqueness of hyperfunction solutions of the Cauchy problems.

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1. Division of hyperfunctions with supports in  $S$ .

Let  $P(x, D)$  be a linear differential operator of order  $m$  with real analytic coefficients defined on a domain  $V$  in  $\mathbb{R}^{n+1}$  and let  $S$  be an oriented real analytic hypersurface in  $V$  non-characteristic with respect to  $P(x, D)$ .

We denote by  $\mathcal{A}$  and  $\mathcal{B}$  ( $'\mathcal{A}$  and  $'\mathcal{B}$ ) the sheaf of real analytic functions and that of hyperfunctions on  $V$  (on  $S$  respectively). When  $K$  is a compact set in  $V$  (in  $S$ ), the space  $\mathcal{A}(K)$  ( $'\mathcal{A}(K)$ ) has a natural (DFS)-topology and its dual is identified with the space  $\mathcal{B}_K(V)$  ( $'\mathcal{B}_K(S)$ ) of hyperfunctions with supports in  $K$  under the inner product

$$\langle \varphi, f \rangle = \int_V \varphi(x) f(x) dx, \quad \varphi \in \mathcal{A}(K), \quad f \in \mathcal{B}_K(V)$$

$$(\langle \varphi, f \rangle = \int_S \varphi(x') f(x') d\omega, \quad \varphi \in '\mathcal{A}(K), \quad f \in '\mathcal{B}_K(S)),$$

where  $dx$  ( $d\omega$ ) denotes the Lebesgue measure on  $V$  (on  $S$ ).

Let  $P'(x, D)$  be the formal dual of  $P(x, D)$ . Then,  $P(x, D)$  and  $P'(x, D)$  induce sheaf homomorphisms  $P(x, D): \mathcal{B} \rightarrow \mathcal{B}$  and  $P'(x, D): \mathcal{A} \rightarrow \mathcal{A}$  respectively. We denote by  $\mathcal{B}^P$  and  $\mathcal{A}^{P'}$  the kernel sheaves, i.e., the sheaf of solutions of

$$(1) \quad P(x, D)f = 0, \quad f \in \mathcal{B},$$

and that of solutions of

$$(2) \quad P'(x, D)\varphi = 0, \quad \varphi \in \mathcal{A}$$

respectively.

Theorem 1. Let  $K$  be a compact set in  $S$ . Then, there is no non-trivial solution of (1) over  $V$  with support in  $K$ :

$$(3) \quad \mathcal{B}_K^P(V) = 0.$$

The quotient space  $\mathcal{B}_K(V)/P\mathcal{B}_K(V)$  is identified with the dual of the (DFS)-space  $\mathcal{A}^{P'}(K)$ .

Proof. Consider the complexes:

$$(4) \quad 0 \longrightarrow \mathcal{A}(K) \xrightarrow{P'(x,D)} \mathcal{A}(K) \longrightarrow 0$$

$$(5) \quad 0 \longleftarrow \mathcal{B}_K(V) \xleftarrow{P(x,D)} \mathcal{B}_K(V) \longleftarrow 0,$$

which are dual to each other in the sense that  $\mathcal{A}(K)$  and  $\mathcal{B}_K(V)$  with their natural (DFS)- and (FS)-topologies are the strong dual spaces of each other and that  $P'(x, D)$  and  $P(x, D)$  are continuous linear operators dual to each other.

The 0-th cohomology group of (4) is  $\mathcal{A}^{P'}(K)$  and the 1-st cohomology group of (4) vanishes by the Cauchy-Kowalevsky theorem. In particular,  $P'(x, D)$  has a closed range. Thus it follows from Serre's lemma (see e.g. [5] Theorem 19) that  $P(x, D)$  has a closed range and that the cohomology groups of (4) and (5) are the strong dual spaces of each other. Therefore,  $\mathcal{B}_K(V)/P\mathcal{B}_K(V)$  is the dual of  $\mathcal{A}^{P'}(K)$  and  $\ker P(x, D) = \mathcal{B}_K^P(V)$  vanishes.

Let  $C_j(x, D)$ ,  $j = 1, 2, \dots, m$ , be linear differential operators of order  $m - j$  with real analytic coefficients on a neighborhood of  $S$  for which  $S$  is non-characteristic (e.g.  $C_j(x, D) = (\partial/\partial n)^{m-j}$ ). Then the Cauchy-Kowalevsky theorem yields the topological isomorphism

$$(6) \quad \rho : \mathcal{A}^{P'}(K) \approx \mathcal{A}(K)^m$$

defined by

$$(7) \quad \mathcal{P}(\varphi) = (C_j(x, D)\varphi|_S), \quad \varphi \in \mathcal{A}^{P'}(K).$$

We have, therefore, the dual isomorphism

$$(8) \quad \mathcal{P}' : {}'\mathcal{B}_K(S)^m \approx \mathcal{B}_K(V)/P\mathcal{B}_K(V).$$

Obviously  $\mathcal{P}$  can be extended by (7) to a continuous linear operator  $\tilde{\mathcal{P}} : \mathcal{A}(K) \rightarrow {}'\mathcal{A}(K)^m$ . Since the open mapping theorem holds for (DFS)-spaces, the exact sequence

$$(9) \quad 0 \rightarrow {}'\mathcal{A}(K)^m \xrightarrow{\mathcal{P}^{-1}} \mathcal{A}(K) \xrightarrow[\text{the}]{P'(x, D)} \mathcal{A}(K) \rightarrow 0$$

splits topologically and we have the topological isomorphism:

$$(10) \quad \mathcal{A}(K) \approx {}'\mathcal{A}(K)^m \oplus \mathcal{A}(K)$$

defined by

$$(11) \quad \varphi \mapsto (C_j(x, D)\varphi|_S) \oplus P'(x, D)\varphi.$$

Correspondingly the dual exact sequence

$$(12) \quad 0 \leftarrow {}'\mathcal{B}_K(S)^m \xleftarrow{(\mathcal{P}^{-1})'} \mathcal{B}_K(V) \xleftarrow{P(x, D)} \mathcal{B}_K(V) \leftarrow 0$$

splits topologically.

Since  $\tilde{\mathcal{P}}$  is the composite of the differential operators  $(C_j(x, D))$  and the restriction to  $S$ , the dual  $\tilde{\mathcal{P}}' : {}'\mathcal{B}_K(S)^m \rightarrow \mathcal{B}_K(V)$  is the mapping  $(f_j) \mapsto \sum_{j=1}^m C'_j(x, D)(f_j \otimes \delta_S)$ , where  $C'_j(x, D)$  is the formal dual of  $C_j(x, D)$  and  $f_j \otimes \delta_S$  is the hyperfunction on  $V$  defined by

$$(13) \quad \langle f_j \otimes \delta_S, \varphi \rangle = \int_S f_j(x') \varphi(x') d\omega, \quad \varphi \in \mathcal{A}(K).$$

Consequently, each  $f \in \mathcal{B}_K(V)$  is uniquely decomposed as

$$(14) \quad f = \sum_{j=1}^m C'_j(x, D)(f_j \otimes \delta_S) + P(x, D)g,$$

where  $f_j \in {}'\mathcal{B}_K(S)$  and  $g \in \mathcal{B}_K(V)$ . Under this correspondence we have a topological isomorphism

$$(15) \quad \mathcal{B}_K(V) \approx {}'\mathcal{B}_K(S)^m \oplus \mathcal{B}_K(V).$$

In particular, the inverse  $(\mathcal{P}')^{-1}: \mathcal{B}_K(V)/P\mathcal{B}_K(V) \approx {}'\mathcal{B}_K(S)^m$  of isomorphism (8) is the mapping which takes the class of  $f$  to  $(f_j)$  in the decomposition (14). Obviously  $f_j$  depend on the choice of  $C_j(x, D)$ . However, the sum  $\sum C_j'(x, D)(f_j \otimes \delta_S)$  and  $P(x, D)g$  do not depend on  $C_j(x, D)$  because neither  $\text{im } \mathcal{P}'^{-1} = \mathcal{A}^{P'}(K)$  nor  $\ker \mathcal{P}'$  depends on  $C_j(x, D)$ .

The uniqueness of the decomposition shows that the components  $f_j$  and  $g$  are independent of the compact set  $K$  which contains the support of  $f$ . Namely we have an isomorphism

$$(16) \quad \Gamma_*(\mathcal{H}_S^0(\mathcal{B})|_S, S) \approx \Gamma_*({}'\mathcal{B}, S)^m \oplus \Gamma_*(\mathcal{H}_S^0(\mathcal{B})|_S, S)$$

which preserves the support, where  $\Gamma_*$  denotes the space of sections with compact supports and  $\mathcal{H}_S^0(\mathcal{B})|_S$  the restriction to  $S$  of the sheaf of sections of  $\mathcal{B}$  with supports in  $S$ .

Let us denote  $\mathcal{H}_S^0(\mathcal{B})|_S$  by  $\mathcal{B}_S$  for short. Since  $\mathcal{B}_S$  and  ${}'\mathcal{B}$  are flabby, it follows that the isomorphism is extended to a sheaf isomorphism (see e.g. [4] Lemma 2.3). Thus we have proved the following theorem.

Theorem 2. If  $C_j'(x, D)$ ,  $j = 1, \dots, m$ , are linear differential operators of order  $m - j$  with real analytic coefficients on a neighborhood of  $S$  for which  $S$  is non-characteristic, then we have a sheaf isomorphism

$$(17) \quad \mathcal{B}_S \approx {}'\mathcal{B}^m \oplus \mathcal{B}_S$$

defined by

$$(18) \quad f = \sum_{j=1}^m C_j'(x, D)(f_j \otimes \delta_S) + P(x, D)g,$$

where  $f \in \mathcal{B}_S$ ,  $f_j \in {}'\mathcal{B}$  and  $g \in \mathcal{B}_S$ . The last component  $g$  does not depend on the choice of  $C_j'(x, D)$ .

In particular, there is no non-trivial solution  $g \in \mathcal{B}^P(V)$  with support in  $S$ :

$$(19) \quad \mathcal{B}_S^P(V) = 0.$$

This theorem means that on division by  $P(x, D)$  each  $f \in \mathcal{B}_S$  has a unique quotient  $g \in \mathcal{B}_S$  and a remainder  $\sum C_j'(x, D)(f_j \otimes \delta_S)$  with  $f_j \in {}'\mathcal{B}$ . We have derived this from the Cauchy-Kowalevsky theorem via the duality of  $\mathcal{A}(K)$  and  $\mathcal{B}_K(V)$  and that of  ${}'\mathcal{A}(K)$  and  ${}'\mathcal{B}_K(S)$ . Conversely Theorem 2 implies the exactness of (12) and hence that of (9). Thus Theorem 2 of division is equivalent to the Cauchy-Kowalevsky theorem.

## 2. Boundary values of hyperfunction solutions.

Let  $W$  be an open subset of  $V$ . We have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow & \mathcal{B}^P(W) & \longrightarrow & \mathcal{B}^P(W \setminus S) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 (20) & 0 \longrightarrow & \mathcal{B}_{S \cap W}(W) & \longrightarrow & \mathcal{B}(W) & \longrightarrow & \mathcal{B}(W \setminus S) \longrightarrow 0 \\
 & \downarrow P(x, D) & & \downarrow P(x, D) & & \downarrow P(x, D) & \\
 & 0 \longrightarrow & \mathcal{B}_{S \cap W}(W) & \longrightarrow & \mathcal{B}(W) & \longrightarrow & \mathcal{B}(W \setminus S) \longrightarrow 0, \\
 & \downarrow & & & & & \\
 & 0 & & & & & 
 \end{array}$$

where  $\mathcal{B}_{S \cap W}(W)$  denotes the space of hyperfunctions on  $W$  with supports in  $S \cap W$ . Since  $\mathcal{B}$  is flabby, the last two rows are exact; the last two columns are exact by the definition; the 0-th cohomology group of the first row and that of the first column vanish since there is no non-trivial solution with support in  $S \cap W$ .

For the remaining cohomology groups we have a natural homomorphism

$$b : \mathcal{B}^P(W \setminus S) / \mathcal{B}^P(W) \longrightarrow \mathcal{B}_{S \cap W}(W) / P \mathcal{B}_{S \cap W}(W).$$

Let  $u \in \mathcal{B}^P(W \setminus S)$  and let  $\tilde{u}$  be an extension in  $\mathcal{B}(W)$ . Since  $P(x, D)\tilde{u} = 0$  on  $W \setminus S$ ,  $P\tilde{u}$  belongs to  $\mathcal{B}_{S \cap W}(W)$ . If  $\tilde{u}_1$  is another extension of  $u$ ,  $\tilde{u} - \tilde{u}_1$  belongs to  $\mathcal{B}_{S \cap W}(W)$ . Therefore the class of  $P\tilde{u}$  in  $\mathcal{B}_{S \cap W}(W) / P \mathcal{B}_{S \cap W}(W)$  is determined uniquely by  $u$ . If  $u$  is the restriction to  $W \setminus S$  of a  $\tilde{u} \in \mathcal{B}^P(W)$ , we have  $P\tilde{u} = 0$ . Thus we can define a homomorphism  $b$  which assigns for the class of  $u \in \mathcal{B}^P(W \setminus S)$  the class of  $P\tilde{u} \in \mathcal{B}_{S \cap W}(W)$ .



Theorem 3. The homomorphism

$$(21) \quad b : \mathcal{B}^P(W \setminus S) / \mathcal{B}^P(W) \longrightarrow \mathcal{B}_{S \cap W}(W) / P \mathcal{B}_{S \cap W}(W).$$

is injective for any open set  $W$  in  $V$  and commutes with restrictions.  $b$  is surjective if and only if

$$(22) \quad P(x, D) \mathcal{B}(W) \supset \mathcal{B}_{S \cap W}(W).$$

Proof. By the definition it is clear that  $b$  commutes with restrictions. To prove the injectivity, let  $P\tilde{u} = Pu_1$  for a  $u_1 \in \mathcal{B}_{S \cap W}(W)$ . Since  $\tilde{u} - u_1 \in \mathcal{B}^P(W)$  and its restriction to  $W \setminus S$  is equal to  $u$ , the class of  $u$  is zero.

Let  $b$  be surjective. Then, for each  $g \in \mathcal{B}_{S \cap W}(W)$  there exist  $h \in \mathcal{B}_{S \cap W}(W)$  and  $\tilde{u} \in \mathcal{B}(W)$  such that  $g + Ph = P\tilde{u}$ . Thus  $\mathcal{B}_{S \cap W}(W) \subset P \mathcal{B}(W)$ .

Conversely suppose that for each  $g \in \mathcal{B}_{S \cap W}(W)$  there is a  $\tilde{u} \in \mathcal{B}(W)$  such that  $g = P\tilde{u}$ . Then, the restriction  $u$  of  $\tilde{u}$  belongs to  $\mathcal{B}^P(W \setminus S)$ . Therefore  $b$  is surjective.

It is known <sup>(that)</sup> (22) holds if the coefficients of  $P(x, D)$  are constants or if  $P(x, D)$  is elliptic.

Now, let  $\omega$  be an open set of  $S$  and let  $W \supset W'$  be two open sets in  $V$  with  $S \cap W = S \cap W' = \omega$ . The restriction  $\mathcal{B}^P(W \setminus S) \longrightarrow \mathcal{B}^P(W' \setminus S)$  induces a homomorphism

$$(23) \quad r : \mathcal{B}^P(W \setminus S) / \mathcal{B}^P(W) \longrightarrow \mathcal{B}^P(W' \setminus S) / \mathcal{B}^P(W').$$

Since  $\mathcal{B}_{S \cap W}(W) / P \mathcal{B}_{S \cap W}(W) = \mathcal{B}_{S \cap W'}(W') / P \mathcal{B}_{S \cap W'}(W')$  and since the injections  $b_W$  and  $b_{W'}$  commute with  $r$ , it follows that  $r$  is injective.

$r$  is surjective if and only if  $\mathcal{B}^P(W' \setminus S) =$

$$\mathcal{B}^P(W \setminus S)|_{W' \setminus S} + \mathcal{B}^P(W')|_{W' \setminus S} \text{ and this holds if}$$

$$(24) \quad H^1(W, \mathcal{B}^P) = 0$$

by the Mayer-Vietoris theorem.

It is also known that (24) holds for any open set  $W$  if the coefficients of  $P(x, D)$  are constants or if  $P(x, D)$  is elliptic.

Taking the inductive limit with respect to the open neighborhoods of  $\omega$ , we have the injection

$$(25) \quad b : (\mathcal{B}_+^P(\omega) \oplus \mathcal{B}_-^P(\omega)) / \mathcal{B}^P(\omega) \longrightarrow \mathcal{B}_S(\omega) / P \mathcal{B}_S(\omega),$$

where  $\mathcal{B}_+^P(\omega)$  ( $\mathcal{B}_-^P(\omega)$ ) denotes the space of germs of solutions on  $W \setminus S$  which vanish on the negative (positive) side of  $S$ .

$\mathcal{B}_\pm^P$  are sheaves over  $S$  which describe the boundary behavior of solutions outside  $S$ .

It follows from Theorem 3 that  $b$  in (25) is surjective if and only if

$$(26) \quad P(x, D) \mathcal{B}(\omega) \supset \mathcal{B}_S(\omega).$$

Furthermore, noticing that the sheaf associated with the presheaf  $(\mathcal{B}_+^P(\omega) \oplus \mathcal{B}_-^P(\omega)) / \mathcal{B}^P(\omega)$  is the restriction  $\mathcal{H}_S^1(\mathcal{B}^P)|_S$  to  $S$  of the first derived sheaf with support in  $S$  (see [4]), we have the injection

$$(27) \quad b : \mathcal{H}_S^1(\mathcal{B}^P)|_S \longrightarrow \mathcal{B}_S / P \mathcal{B}_S$$

which is surjective if and only if

$$(28) \quad P(x, D) \mathcal{B}|_S \supset \mathcal{B}_S.$$

Obviously (28) is satisfied if  $P(x, D)$  is locally solvable on  $S$  i.e. if

$$(29) \quad P(x, D) : \mathcal{B}(x) \longrightarrow \mathcal{B}(x) \text{ is surjective for } x \in S.$$

This is known for operators with constant coefficients or of elliptic type. Moreover, T. Kawai [3] proves the existence of local elementary solutions and hence the local solvability of operators  $P(x, D)$  of simple characteristics with real principal parts. Thus (27) is an isomorphism for such operators. Combining this with the isomorphism  $\mathcal{B}_S/P\mathcal{B}_S \approx {}'\mathcal{B}^m$  given in Theorem 2, we have an isomorphism

$$(30) \quad \mathcal{H}_S^1(\mathcal{B}^P)|_S \approx {}'\mathcal{B}^m.$$

Definition. Let  $W$  be an open set in  $V$ , let  $\omega = S \cap W$  and let  $W_+$  be the positive part of  $W \setminus S$ . For each solution  $u \in \mathcal{B}^P(W_+)$  we define its boundary values  $(f_j) \in {}'\mathcal{B}(\omega)^m$  on  $S$  to be the image of  $u$  under the composite of mappings  $\mathcal{B}^P(W_+) \rightarrow \mathcal{B}_+^P(\omega) \rightarrow (\mathcal{B}_+^P(\omega) \oplus \mathcal{B}_-^P(\omega))/\mathcal{B}^P(\omega) \xrightarrow{b} \mathcal{B}_S(\omega)/P\mathcal{B}_S(\omega) \rightarrow {}'\mathcal{B}(\omega)^m$ , where the last mapping is the isomorphism obtained in Theorem 2 as an extension of  $(\rho')^{-1}$  in (8). In other words,  $(f_j) \in {}'\mathcal{B}(\omega)^m$  is the unique  $m$ -tuple of hyperfunctions on  $\omega$  which satisfy

$$(31) \quad P(x, D)\tilde{u} = \sum_{j=1}^m C_j'(x, D)(f_j \otimes \delta_S)$$

for an extension  $\tilde{u} \in \mathcal{B}(W)$  vanishing on the negative side of  $W \setminus S$ .

As we remarked earlier, the extension  $\tilde{u}$  which satisfies (31) is uniquely determined by  $u$  and does not depend on the choice of  $C_j(x, D)$ , so that we call  $\tilde{u}$  the canonical extension of  $u$ .

Let  $\theta_S$  be the characteristic function of  $W_+$  in  $W$ . Then, there are unique linear differential operators  $B_j(x, D)$ ,  $j = 1, \dots, m$ , of order  $j-1$  with real analytic coefficients in a neighborhood of  $S$  such that  $S$  is non-characteristic and that

$$\begin{aligned}
& P(x, D)(\theta_S(x)u(x)) - \theta_S(x)(P(x, D)u(x)) \\
(32) \quad &= \sum_{j=1}^m C'_j(x, D)(B_j(x, D)u(x)(1 \otimes \delta_S)) \\
&= \sum_{j=1}^m C'_j(x, D)((B_j(x, D)u(x))|_S \otimes \delta_S)
\end{aligned}$$

for any  $u \in \mathcal{Q}(W)$  or more generally for any  $u \in \mathcal{B}(W)$  which is real analytic in the normal direction on  $S$  (see [1] for the real analyticity in parameter and the restrictions of hyperfunctions to submanifolds).

Conversely if  $B_j(x, D)$ ,  $j = 1, \dots, m$ , are linear differential operators of order  $j-1$  with real analytic coefficients for which  $S$  is non-characteristic, we can find linear differential operators  $C_j(x, D)$  of order  $m-j$  such that  $S$  is non-characteristic and that (32) holds. This is only a local formulation of Green's formula.

Therefore, if  $u$  is the restriction to  $W_+$  of a solution  $u_1 \in \mathcal{Q}^P(W)$  we have

$$(33) \quad f_j = B_j(x, D)u_1|_S, \quad j = 1, \dots, m.$$

This holds also for the restriction  $u$  of a solution  $u_1 \in \mathcal{B}^P(W)$ , because  $u_1$  is real analytic in the normal direction on  $S$  by Sato's fundamental theorem of analyticity (see [1]).

Taking this into account we will write the boundary values

$$(34) \quad f_j = B_j(x, D)u|_{S_+}, \quad j = 1, \dots, m.$$

Similarly we can define the boundary values  $B_j(x, D)u|_{S_-}$  of solutions  $u$  on the negative side of  $W \setminus S$ . The following is clear from the definition.

Theorem 4. A solution  $u \in \mathcal{B}^P(W \setminus S)$  is extended to a solution  $u \in \mathcal{B}^P(W)$  if and only if

$$(35) \quad B_j(x, D)u|_{S_+} = B_j(x, D)u|_{S_-}, \quad j = 1, \dots, m.$$

This may be regarded as a generalization of the classical Painlevé theorem.

If the operator  $P(x, D)$  is locally solvable on  $S$  or if (28) holds, then the isomorphism (30) shows that the Plemelj problem

$$(36) \quad B_j(x, D)u|_{S_+} - B_j(x, D)u|_{S_-} = f_j, \quad j = 1, \dots, m$$

has a local solution  $u \in \mathcal{B}_+^P(x) \oplus \mathcal{B}_-^P(x)$  for any  $f_j \in {}'\mathcal{B}(x)$  on  $S$ .

[21]

Lastly the Holmgren theorem by T. Kawai<sup>Λ</sup> and P. Schapira<sup>[8]</sup><sub>7</sub> asserts that

$$(37) \quad \mathcal{B}_+^P(\omega) \cap \mathcal{B}^P(\omega) = \{0\} \quad \text{and} \quad \mathcal{B}_-^P(\omega) \cap \mathcal{B}^P(\omega) = \{0\}.$$

Therefore the mapping  $\mathcal{B}_+^P(\omega) \rightarrow {}'\mathcal{B}(\omega)^m$  is injective.

Thus we have

Theorem 5. A solution  $u \in \mathcal{B}^P(W_+)$  on the positive side of  $W \setminus S$  vanishes in a neighborhood of  $\omega = W \cap S$  if and only if the boundary values  $B_j(x, D)u|_S$  vanish for all  $j = 1, \dots, m$ .

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